A variational approach to the regularity theory for optimal transportation

In this mini-course, we shall explain the variational approach to regularity theory for optimal transportation introduced in [8]. This approach does completely bypass the celebrated regularity theory of Caffarelli [2], which is based on the regularity theory for the Monge-Ampère equation as a fully nonlinear elliptic equation with a comparison principle. Nonetheless, one recovers the same partial regularity theory [5, 4].

The advantage of the variational approach resides in its robustness regarding the regularity of the measures, which can be arbitrary measures [7][Theorem 1.4], and in terms of the problem formulation, e. g. by its extension to almost minimizers [10]. The former for instance is crucial in order to tackle the widely popular matching problem [3, 1]. e. g. the optimal transportation between (random) point clouds, as carried out in [7, 6, 9]. The latter is convenient when treating more general than square Euclidean cost functions.

The variational approach follows de Giorgi's philosophy for minimal surfaces. At its core is the approximation of the displacement by the gradient of a harmonic function. This approximation is based on the Eulerian formulation of optimal transportation, which reveals its strict convexity and the proximity to the H^{-1}-norm. In this mini-course, we shall give a pretty self-contained derivation of this harmonic approximation result, and establish applications to the matching problem.

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